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STABILITY MARGINS FOR DISCRETE-TIME OPTIMAL REGULATORS

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ABSTRACT

A simple equation for guaranteed gain and phase margins for LQ state feedback in discrete-time systems will be presented. The results are based on the feedback gain and are made for SISO systems. An extension to MIMO systems are also showed. These new equations can also be used to find guaranteed gain and phase margins for optimal eigenvalue design methods.

INTRODUCTION

It is well known that the excellent stability margins of the LQ regulator in continuous-time don't exist in discrete-time [1]. Therefore equations for gain and phase margins have been derived for SISO systems [2] and MIMO systems, see e.g. Safonov [1] and Shaked [3]. However, some of these guaranteed stability margins are very difficult to find, and some of them are also conservative. An alternative to LQ, is to use another state-feedback regulator, where the stability margins are well defined. Tsu-Tian Lee et. al. [4] have presented such a state-feedback regulator with specified stability characteristics. In this short paper simple equations for the discrete-time LQ regulators stability margins will be found. The results are entirely based on the systems matrices and the LQ feedback matrix, and they are derived from Safonov's conic sector stability results [1]. It will be shown that these new results have more general applications than Safonov's previous results.

PREVIOUS RESULTS

Consider a discrete-time linear system described by:

$$x(t+1) = Ax(t) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) \quad (2)$$

where $x(t)$ is a n -dim. state vector, $u(t)$ is a m -dim. control input and $y(t)$ is a p -dim. output vector. We shall assume that (A,B) is stabilizable and (A,C) is detectable. A linear quadratic state feedback controller (LQSF) is given by, see f.ex. [5]:

$$u(t) = -Gx(t) \quad (3)$$

where G is given by the following equations:

$$\text{Index } J = \sum_{t=0}^{\infty} (x^T(t)Qx(t) + u^T(t)Ru(t)) \quad (4)$$

where Q is a non-negative-definite symmetric matrix and R is a positive-definite symmetric matrix.

Riccati equation:

$$K = A^T KA + Q - A^T KB(R + B^T KB)^{-1} B^T KA \quad (5)$$

Feedback gain:

$$G = (R + B^T KB)^{-1} B^T KA = FA \quad (6)$$

Safonov [1] has shown that the LQ-regulator design guarantee stability margins if $R = \text{diag}(r_1, \dots, r_m)$.

Gain margins k_i :

$$\frac{1}{1 + a_i} < k_i < \frac{1}{1 - a_i} \quad (7)$$

Phase margins ϕ_i :

$$|\phi_i| < 2\arcsin(a_i/2) \quad (8)$$

where a_i is given by:

$$a_i = \sqrt{\frac{r_i}{r_i + \lambda_{\max}(B^T KB)}} \quad (9)$$

The equations for gain- and phase-margins are based on Zames' conic sector theory [6].

NEW RESULTS

For SISO systems eq.(9) can be reduced to a much simpler form, and the gain and the phase margins are given by the following:

Theorem 1.

Gain and phase margins for LQ-regulators in SISO systems are given by eq.(7) and eq.(8) where a is given by:

$$a = \sqrt{1 - FB} \quad (10)$$

For MIMO systems eq.(9) can also be reduced if the input weight matrix can be written $R = \rho I$, where I is the unit matrix. The guaranteed gain and phase margins will be the same in all channels, $a = a_i$ for $i=1, \dots, m$.

For the MIMO case, the next theorem apply:

Theorem 2.

If $R = \rho I$, gain and phase margins for multivariable LQ-regulator are given by eq.(7) and (8) where $a_i = a$, $i = 1, \dots, m$:

$$a = \sqrt{1 - \lambda_{\max}(FB)} \quad (11)$$

Proof of eq.(10) and (11) are in app. A, where the equations for a_i is given when $R = \text{diag}(r_1, \dots, r_m)$.

There exist dual results for optimal observer design.

In this section some consequences of theorem 1

and 2 are discussed.

- * The freedom of selecting the input weight matrix $R = \rho I$ is limited when eq.(11) is used on a multivariable LQ-regulator. However, by manipulating the state weight matrix Q we can get the same closed-loop eigenvalues as for the original weight matrices (Q, R) , see Solheim [7].
- * It is possible to make the stability margins larger by increasing the input weight matrix $R = \rho I$, which is a consequence of eq.(5) and (9).
- * It is possible to use theorem 1 and 2 to find guaranteed stability margins for gains which are a combination of LQ-gains, $F = F_1 + F_2$ if the input weight matrices are given by $R_1 = \rho_1 I$, $R_2 = \rho_2 I$ or $R_1 = R_2 = \rho I$. This couldn't be done by using Safonov's previous results, because we can't generally find the resulting Riccati solution for F which is necessary. By this fact it is possible to find guaranteed stability margins for gains designed by others methods, if we know that the gains also are LQ-gains and for MIMO systems also the input weight matrix $R = \rho I$ or $\text{diag}(r_1, \dots, r_m)$, (LQ-gains satisfy the following: FB is a symmetric matrix and $0 < \lambda(FB) < 1$). An example of such a design method is Solheim's useful optimal eigenvalue design method for MIMO systems [7].

THE RICCATI SOLUTION

As a consequence of the previous results in this paper, it is possible to find the resulting Riccati solution for $F = F_1 + F_2$ in the SISO case, when F_1, F_2 are LQ-gains.

Let F be given by eq.(12):

$$F = F_1 + F_2 \quad (12)$$

and F, F_1, F_2 are given by eq.(6).

Now let α_i be given by:

$$\alpha_i = (R + B^T K B)^{-1} \quad (13)$$

Eq. (13) can be rewritten as:

$$\alpha_i = R^{-1} (I - FB)$$

It is now easy to see that eq.(12) together with eq.(6) gives:

$$\alpha B^T K = \alpha_1 B^T K_1 + \alpha_2 B^T K_2 \quad (14)$$

or

$$B^T K = \alpha^{-1} \alpha_1 B^T K_1 + \alpha^{-1} \alpha_2 B^T K_2$$

In the special case with $\text{rank}(B) = n$ the resulting Riccati solution is given by:

$$K = B^{-T} \alpha^{-1} (\alpha_1 B^T K_1 + \alpha_2 B^T K_2) \quad (15)$$

In the SISO case eq.(14) can be rewritten:

$$B^T (K - \alpha^{-1} \alpha_1 K_1 - \alpha^{-1} \alpha_2 K_2) = 0 \quad (16)$$

This equation is satisfied if K is chosen as:

$$K = \alpha^{-1} \alpha_1 K_1 + \alpha^{-1} \alpha_2 K_2 \quad (17)$$

Note that in eq.(15) and (17) the input weight matrix R for the combination gain F is free to choose. The Riccati solution K will of course

depend of this choice.

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APPENDIX A.

Proof of eq. (10) and (11).

From eq. (6):

$$F = (R + B^T K B)^{-1} B^T K \quad (6)$$

$$(R + B^T K B) F B = B^T K B \quad (A1)$$

$$B^T K B = R F B (I - FB)^{-1} \quad (A2)$$

$$a_i^2 = \frac{1}{1 + r_i^{-1} \lambda_{\max}(B^T K B)} \quad (A3)$$

Eq. (A2) + (A3):

$$a_i^2 = \frac{1}{1 + r_i^{-1} \lambda_{\max}(R F B (I - FB)^{-1})} \quad (A4)$$

$$= \lambda_{\max}^{-1} (I + r_i^{-1} R F B (I - FB)^{-1}) \quad (A5)$$

$$= \lambda_{\max}^{-1} ((I - FB) + r_i^{-1} R F B) (I - FB)^{-1} \quad (A6)$$

$$= \lambda_{\max}^{-1} ((I + \alpha_i F B) (I - FB)^{-1}) \quad (A7)$$

where $\alpha_i = r_i^{-1} R - I$

If $R = \rho I$ then $\alpha_i = \rho^{-1} \rho I - I = 0$, and eq. (A7) can be reduced to:

$$a_i = a, \quad i = 1, \dots, m:$$

$$a^2 = \lambda_{\max}^{-1} ((I - FB)^{-1}) \quad (A8)$$

$$= \lambda_{\max} (I - FB) \quad \text{for MIMO systems} \quad (A9)$$

$$= 1 - FB \quad \text{for SISO systems} \quad (A10)$$